Linear Spherical Sliced Optimal Transport: A Fast Metric for Comparing Spherical Data

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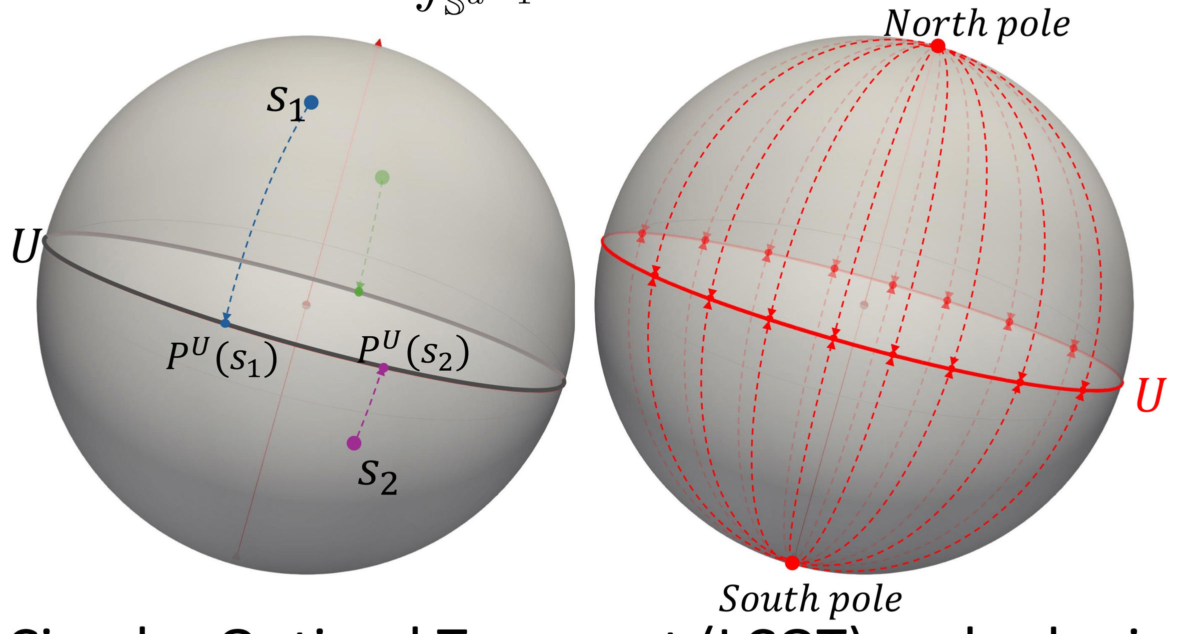
Summary

We propose to embed spherical distributions into an L^2 space via Linear Spherical Sliced Optimal Transport (LSSOT). The LSSOT embedding induces a novel spherical metric and enhances the efficiency of group analysis in spherical domains.

Background

- Spherical Radon Transform $\mathcal{R}$ projects spherical distributions on great circles. For a density function $f \in L^1(\mathbb{S}^{d-1})$, and a rank 2 projector $U \in V_2(\mathbb{R}^d)$ that intersects $\mathbb{S}^{d-1}$ at a great circle,

$$\mathcal{R}f(U,z) := \int_{\mathbb{S}^{d-1}} \delta(P^U(s) = z) f(s) ds$$



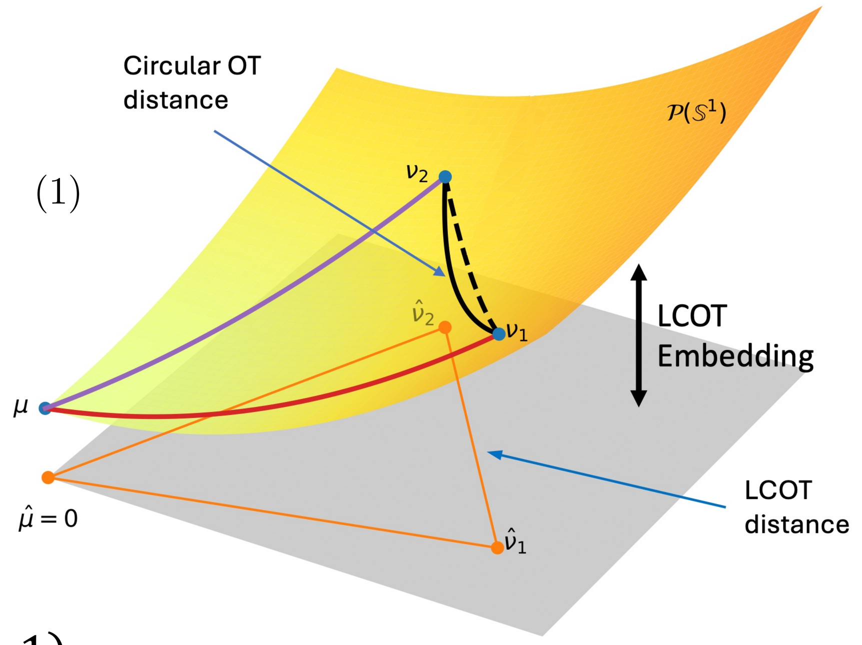
- Linear Circular Optimal Transport (LCOT) embeds circular distributions into an L^2 space and induces a metric on $\mathcal{P}(\mathbb{S}^1)$.

Circular Optimal Transport:

$$COT_h(\mu,\nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{S}^1 \times \mathbb{S}^1} h(|x-y|_{\mathbb{S}^1}) d\gamma(x,y)$$
$$= \inf_{\alpha \in \mathbb{R}} \int_0^1 h(|F_\mu^{-1}(x) - F_\nu^{-1}(x-\alpha)|_{\mathbb{R}})\, dx \quad (1)$$

Linear Circular Optimal Transport:

1. Fix the reference measure $\mu = Uniform(\mathbb{S}^1)$, and let $h(x) = |x|^2$;
2. The optimal $\alpha^* = \mathbb{E}(\nu) - 1/2$ in Eq. (1)
3. The LCOT embedding is defined as $\hat{\nu}(x) := F_\nu^{-1}(F_\mu(x) - \alpha^*) - x, \forall x \in [0,1)$
4. The LCOT metric: $LCOT_2(\nu_1,\nu_2) = ||\hat{\nu}_1 - \hat{\nu}_2||_{L^2(\mathbb{S}^1)}$



LSSOT Embedding and Metric

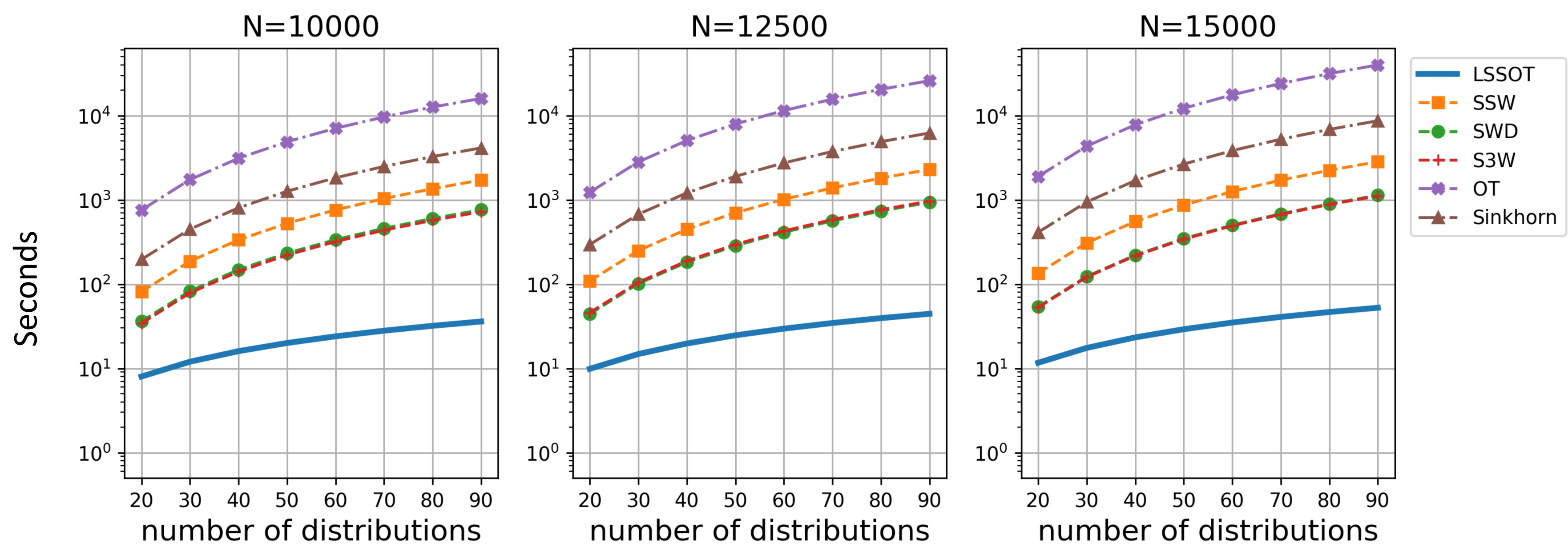
- Fix the reference measure $\mu = Uniform(\mathbb{S}^{d-1})$;
- The LSSOT Embedding for $\nu_i \in \mathcal{P}(\mathbb{S}^{d-1})$:

$$\widehat{\nu_i}^S(x,U) := F^{-1}_{P^U_\# \nu_i}\left(F_{P^U_\# \mu}(x) - \mathbb{E}(P^U_\# \nu_i) + \frac{1}{2}\right) - x, \quad \forall x \in \mathbb{S}^{d-1}$$

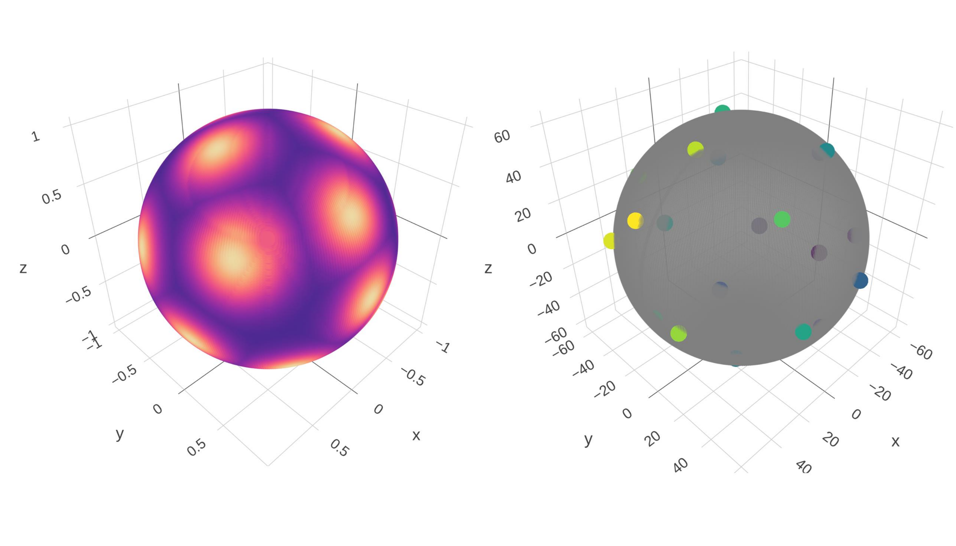
- The LSSOT metric:

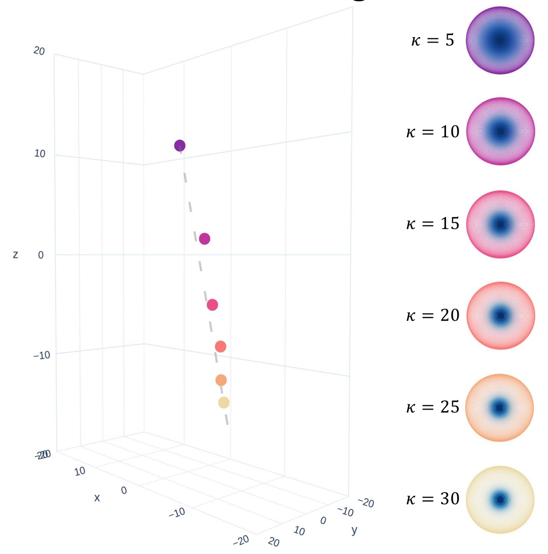
$$(LSSOT_2(\nu_1,\nu_2))^2 := \int_{V_2(\mathbb{R}^d)} (LCOT_2(P^U_\# \nu_1, P^U_\# \nu_2))^2 \, d\sigma(U)$$
$$= \int_{V_2(\mathbb{R}^d)} \|\widehat{\nu_1}^S(\cdot,U) - \widehat{\nu_2}^S(\cdot,U)\|^2_{L^2(\mathbb{S}^1)} \, d\sigma(U)$$

Computation Efficiency

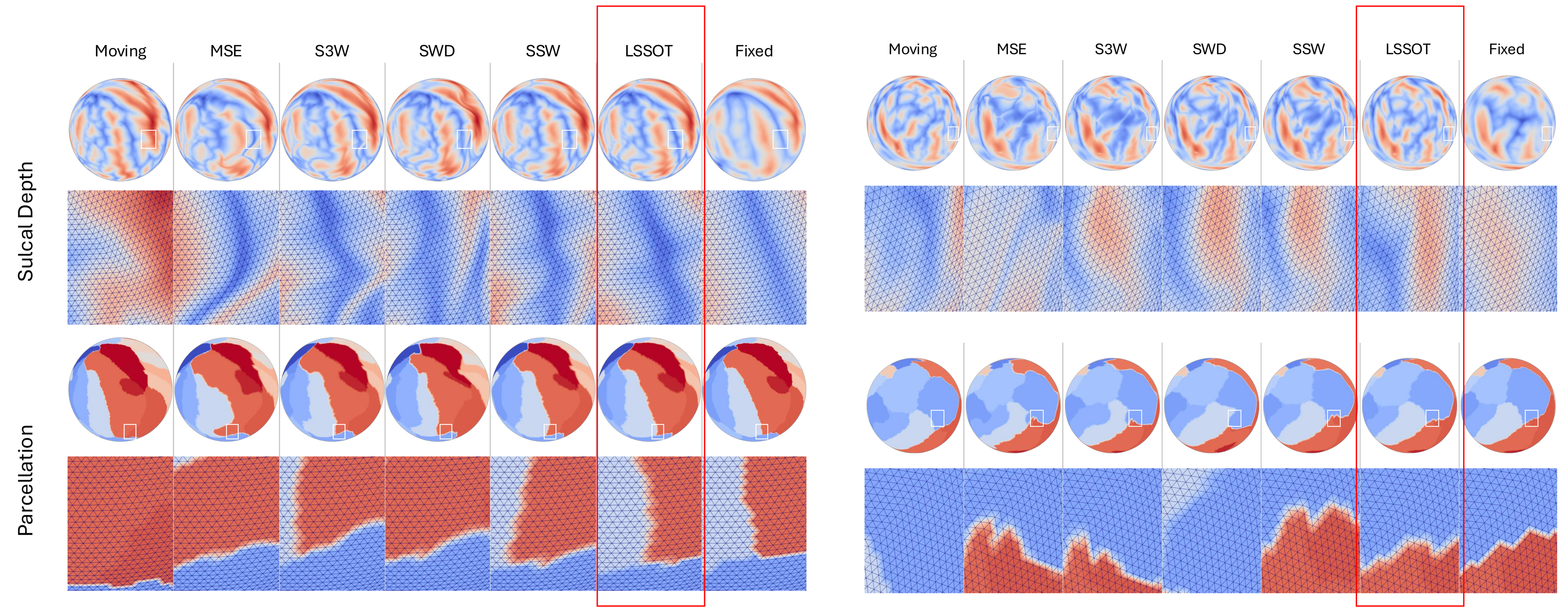


Rotation and Scaling Geometry





Cortical Surface Registration



subject-A00065749 (NKI) subject-A00076798 (NKI)

		Freesurfer	MSE	S3W	SWD	SSW	LSSOT (Ours)
NKI (Left Hemisphere)	LSSOT(↓)	0.2877±0.0392	0.2754±0.0611	0.2298±0.0346	0.2411± 0.0366	**0.2079±0.0369**	**0.1890±0.0361**
	SWD(↓)	0.0060±0.0020	**0.0051±0.0017**	0.0053±0.0010	**0.0027±0.0011**	0.0059±0.0011	0.0052±0.0011
	MAE(↓)	**0.1053±0.0214**	**0.1129±0.0471**	0.2278±0.0372	0.2658±0.0410	0.2145±0.0266	0.2516±0.0397
	CC(↑)	0.8190±0.0264	0.8269±0.0425	**0.9216±0.0174**	**0.8722±0.0302**	0.8671±0.0314	0.8649±0.0295
	Dice (↑)	0.7541±0.0651**	0.7746±0.0861**	**0.8498±0.0670****	0.7984±0.0539**	0.8429±0.0692*	**0.8462±0.0548**
	Edge Dist.(↓)	0.3207±0.0436**	0.3060±0.0404**	0.3922±0.0647**	0.3442±0.0346**	**0.2476±0.0348****	**0.2365±0.0343**
	Area Dist.(↓)	0.4652±0.0698**	0.4305±0.0488**	0.4048±0.0752**	0.4073±0.0368**	**0.2733±0.0402****	**0.2897±0.0396**
	Time(seconds)(↓)	—	**73.07**	121.00	118.96	1350.96	**101.01**

Point Cloud Interpolation